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A Study on Fractional Fourier Series of Four Types of Matrix Fractional Functions

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Abstract: **In this paper, based on a new multiplication of fractional analytic functions, we find the fractional Fourier series of four types of matrix fractional functions. Matrix fractional Euler's formula and matrix fractional DeMoivre's formula play important roles in this article. In fact, our results are generalizations of ordinary calculus results.**

Keywords: **New multiplication, fractional analytic functions, fractional Fourier series, matrix fractional functions.**

I. INTRODUCTION

In recent decades, the applications of fractional calculus in various fields of science is growing rapidly, such as physics, biology, mechanics, electrical engineering, viscoelasticity, control theory, modelling, economics, etc [1-15]. However, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivative. Other useful definitions include Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie type of R-L fractional derivative to avoid non-zero fractional derivative of constant function [16-20].

In this paper, based on a new multiplication of fractional analytic functions, we use some methods to obtain the Fourier series expansions of the following four types of matrix fractional functions:

$$sin_\alpha\big(rcos_\alpha(tAx^\alpha)\big)\otimes_\alpha cosh_\alpha\big(rsin_\alpha(tAx^\alpha)\big),$$

$$cos_\alpha\big(rcos_\alpha(tAx^\alpha)\big)\otimes_\alpha sinh_\alpha\big(rsin_\alpha(tAx^\alpha)\big),$$

$$cos_\alpha\big(rcos_\alpha(tAx^\alpha)\big)\otimes_\alpha cosh_\alpha\big(rsin_\alpha(tAx^\alpha)\big),$$

$$sin_\alpha\big(rcos_\alpha(tAx^\alpha)\big)\otimes_\alpha sinh_\alpha\big(rsin_\alpha(tAx^\alpha)\big),$$

where $0 < \alpha \leq 1$, r, t are real numbers, and A is a real matrix. Matrix fractional Euler's formula and matrix fractional DeMoivre's formula play important roles in this article. In fact, our results are generalizations of traditional calculus results.

II. PRELIMINARIES

At first, we introduce the definition of fractional analytic function.

Definition 2.1 ([21])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

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In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.2 ([22])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{1}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\,. \tag{2}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}.\end{aligned} \tag{3}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}.\end{aligned} \tag{4}$$

Definition 2.3 ([23])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{6}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\big(g_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{7}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\big(f_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{8}$$

Definition 2.4 ([24])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\big(g_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\big(f_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{12}$$

Definition 2.5 ([25])**:** If $0 < \alpha \leq 1$, t is a real number, and A is a matrix. The matrix α-fractional exponential function, matrix α-fractional cosine function, and matrix α-fractional sine function are defined as follows:

$$E_\alpha(tAx^\alpha) = \sum_{n=0}^{\infty}(tA)^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(tA\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}, \tag{13}$$

$$cos_\alpha(tAx^\alpha) = \sum_{n=0}^{\infty}(tA)^{2n} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(tA\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{14}$$

$$sin_\alpha(tAx^\alpha) = \sum_{n=0}^{\infty}(tA)^{2n+1} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(tA\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{15}$$

In addition, the matrix α-fractional hyperbolic cosine function and matrix α-fractional hyperbolic sine function are defined as follows:

$$cosh_\alpha(tAx^\alpha) = \frac{1}{2}[E_\alpha(tAx^\alpha) + E_\alpha(-tAx^\alpha)] = \sum_{n=0}^{\infty}(tA)^{2n} \frac{x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(tA\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{16}$$

and

$$sinh_\alpha(tAx^\alpha) = \frac{1}{2}[E_\alpha(tAx^\alpha) - E_\alpha(-tAx^\alpha)] = \sum_{n=0}^{\infty}(tA)^{2n+1} \frac{x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(tA\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{17}$$

Theorem 2.6 (matrix fractional Euler's formula)([26])**:** *If* $0 < \alpha \leq 1$, $i = \sqrt{-1}$, *and A is a real matrix, then*

$$E_\alpha(iAx^\alpha) = cos_\alpha(Ax^\alpha) + isin_\alpha(Ax^\alpha). \tag{18}$$

Theorem 2.7 (matrix fractional DeMoivre's formula)([27])**:** *If* $0 < \alpha \leq 1$, *p is an integer, and A is a real matrix, then*

$$[cos_\alpha(Ax^\alpha) + isin_\alpha(Ax^\alpha)]^{\otimes_\alpha p} = cos_\alpha(pAx^\alpha) + isin_\alpha(pAx^\alpha). \tag{19}$$

III. MAIN RESULTS

In this section, we obtain the Fourier series expansions of four types of matrix fractional functions by using some methods.

Theorem 3.1: *Suppose that* $0 < \alpha \leq 1$, *r, t are real numbers, and A is a real matrix. Then*

$$sin_\alpha\left(rcos_\alpha(tAx^\alpha)\right)\otimes_\alpha cosh_\alpha\left(rsin_\alpha(tAx^\alpha)\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}r^{2n+1}cos_\alpha((2n+1)tAx^\alpha), \tag{20}$$

$$cos_\alpha\left(rcos_\alpha(tAx^\alpha)\right)\otimes_\alpha sinh_\alpha\left(rsin_\alpha(tAx^\alpha)\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}r^{2n+1}sin_\alpha((2n+1)tAx^\alpha), \tag{21}$$

$$cos_\alpha\left(rcos_\alpha(tAx^\alpha)\right)\otimes_\alpha cosh_\alpha\left(rsin_\alpha(tAx^\alpha)\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}r^{2n}cos_\alpha(2ntAx^\alpha), \tag{22}$$

$$sin_\alpha\left(rcos_\alpha(tAx^\alpha)\right)\otimes_\alpha sinh_\alpha\left(rsin_\alpha(tAx^\alpha)\right) = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}r^{2n}sin_\alpha(2ntAx^\alpha). \tag{23}$$

Proof Since

$$\begin{aligned} & sin_\alpha\left(rE_\alpha(itAx^\alpha)\right) \\ &= sin_\alpha\left(rcos_\alpha(tAx^\alpha) + i\cdot rsin_\alpha(tAx^\alpha)\right) \\ &= sin_\alpha\left(rcos_\alpha(tAx^\alpha)\right)\otimes_\alpha cosh_\alpha\left(rsin_\alpha(tAx^\alpha)\right) + i\cdot cos_\alpha\left(rcos_\alpha(tAx^\alpha)\right)\otimes_\alpha sinh_\alpha\left(rsin_\alpha(tAx^\alpha)\right). \end{aligned} \tag{24}$$

And

$$\begin{aligned} & sin_\alpha\left(rE_\alpha(itAx^\alpha)\right) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(rE_\alpha(itAx^\alpha)\right)^{\otimes_\alpha (2n+1)} \quad \text{(by Eq. (15))} \end{aligned}$$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} r^{2n+1} E_\alpha(i(2n+1)tAx^\alpha)$ (by matrix fractional DeMoivre's formula)

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} r^{2n+1} [cos_\alpha((2n+1)tAx^\alpha) + i \cdot sin_\alpha((2n+1)tAx^\alpha)]$. (by matrix fractional Euler's formula)

It follows that

$$sin_\alpha\big(rcos_\alpha(tAx^\alpha)\big) \otimes_\alpha cosh_\alpha\big(rsin_\alpha(tAx^\alpha)\big) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} r^{2n+1} cos_\alpha((2n+1)tAx^\alpha) \,,$$

and

$$cos_\alpha\big(rcos_\alpha(tAx^\alpha)\big) \otimes_\alpha sinh_\alpha\big(rsin_\alpha(tAx^\alpha)\big) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} r^{2n+1} sin_\alpha((2n+1)tAx^\alpha) \,.$$

On the other hand, since

$cos_\alpha\big(rE_\alpha(itAx^\alpha)\big)$

$= cos_\alpha\big(rcos_\alpha(tAx^\alpha) + i \cdot rsin_\alpha(tAx^\alpha)\big)$

$= cos_\alpha\big(rcos_\alpha(tAx^\alpha)\big) \otimes_\alpha cosh_\alpha\big(rsin_\alpha(tAx^\alpha)\big) - i \cdot sin_\alpha\big(rcos_\alpha(tAx^\alpha)\big) \otimes_\alpha sinh_\alpha\big(rsin_\alpha(tAx^\alpha)\big)$. (25)

And

$cos_\alpha\big(rE_\alpha(itAx^\alpha)\big)$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \big(rE_\alpha(itAx^\alpha)\big)^{\otimes_\alpha 2n}$ (by Eq. (14))

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} r^{2n} E_\alpha(i2ntAx^\alpha)$ (by matrix fractional DeMoivre's formula)

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} r^{2n} [cos_\alpha(2ntAx^\alpha) + i \cdot sin_\alpha(2ntAx^\alpha)]$. (by matrix fractional Euler's formula)

It follows that

$$cos_\alpha\big(rcos_\alpha(tAx^\alpha)\big) \otimes_\alpha cosh_\alpha\big(rsin_\alpha(tAx^\alpha)\big) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} r^{2n} cos_\alpha(2ntAx^\alpha) \,,$$

and

$$sin_\alpha\big(rcos_\alpha(tAx^\alpha)\big) \otimes_\alpha sinh_\alpha\big(rsin_\alpha(tAx^\alpha)\big) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} r^{2n} sin_\alpha(2ntAx^\alpha) \,.$$ q.e.d.

IV. CONCLUSION

In this paper, we obtain the fractional Fourier series of four types of matrix fractional functions based on a new multiplication of fractional analytic functions. Matrix fractional Euler's formula and matrix fractional DeMoivre's formula play important roles in this article. In fact, our results are generalizations of classical calculus results. In the future, we will continue to study the problems in applied mathematics and fractional differential equations.

REFERENCES

[1] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp, 41-45, 2016.

[2] R. L. Magin, Fractional calculus models of complex dynamics in biological tissues, Computers & Mathematics with Applications, vol. 59, no. 5, pp. 1586-1593, 2010.

[3] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[4] F. C. Meral, T. J. Royston, R. Magin, Fractional calculus in viscoelasticity: an experimental study, Communications in Nonlinear Science and Numerical Simulation, vol. 15, no. 4, pp. 939-945, 2010.

[5] R. Hilfer (ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[6] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers, Vol. 1, Background and Theory, Vol. 2, Application. Springer, 2013.

[7] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, Vol. 8, No. 5, 660, 2020.

[8] M. F. Silva, J. A. T. Machado, A. M. Lopes, Fractional order control of a hexapod robot, Nonlinear Dynamics, vol. 38, pp. 417-433, 2004.

[9] A. Carpinteri, F. Mainardi, (Eds.), Fractals and fractional calculus in continuum mechanics, Springer, Wien, 1997.

[10] N. Heymans, Dynamic measurements in long-memory materials: fractional calculus evaluation of approach to steady state, Journal of Vibration and Control, vol. 14, no. 9, pp. 1587-1596, 2008.

[11] R. C. Koeller, Applications of fractional calculus to the theory of viscoelasticity, Journal of Applied Mechanics, vol. 51, no. 2, 299, 1984.

[12] T. Sandev, R. Metzler, & Ž. Tomovski, Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative, Journal of Physics A: Mathematical and Theoretical, vol. 44, no. 25, 255203, 2011.

[13] J. P. Yan, C. P. Li, On chaos synchronization of fractional differential equations, Chaos, Solitons & Fractals, vol. 32, pp. 725-735, 2007.

[14] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[15] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[16] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations; John Willy and Sons, Inc.: New York, NY, USA, 1993.

[17] K. B. Oldham, J. Spanier, The Fractional Calculus; Academic Press: New York, NY, USA, 1974.

[18] I. Podlubny, Fractional Differential Equations; Academic Press: New York, NY, USA, 1999.

[19] S. Das, Functional Fractional Calculus, 2nd Edition, Springer-Verlag, 2011.

[20] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[21] C. -H, Yu, Evaluating fractional derivatives of two matrix fractional functions based on Jumarie type of Riemann-Liouville fractional derivative, International Journal of Engineering Research and Reviews, vol. 12, no. 4, pp. 39-43, 2024.

[22] C. -H, Yu, Studying three types of matrix fractional integrals, International Journal of Interdisciplinary Research and Innovations, vol. 12, no. 4, pp. 35-39, 2024.

[23] C. -H, Yu, Fractional partial differential problem of some matrix two-variables fractional functions, International Journal of Mechanical and Industrial Technology, vol. 12, no. 2, pp. 6-13, 2024.

[24] C. -H, Yu, Study of two matrix fractional integrals by using differentiation under fractional integral sign, International Journal of Civil and Structural Engineering Research, vol. 12, no. 2, pp. 24-30, 2024.

[25] C. -H, Yu, Study of some type of matrix fractional integral, International Journal of Recent Research in Civil and Mechanical Engineering, vol. 11, no. 2, pp. 5-8, 2024.

[26] C. -H, Yu, Evaluating fractional derivatives of two matrix fractional functions based on Jumarie type of Riemann-Liouville fractional derivative, International Journal of Engineering Research and Reviews, vol. 12, no. 4, pp. 39-43, 2024.

[27] C. -H, Yu, Studying three types of matrix fractional integrals, International Journal of Interdisciplinary Research and Innovations, vol. 12, no. 4, pp. 35-39, 2024.